The Hyperbolic Functions

6.2

Introduction

The hyperbolic functions \( \sinh x \), \( \cosh x \), \( \tanh x \) etc are certain combinations of the exponential functions \( e^x \) and \( e^{-x} \). The notation implies a close relationship between these functions and the trigonometric functions \( \sin x \), \( \cos x \), \( \tan x \) etc. The close relationship is algebraic rather than geometrical. For example, the functions \( \cosh x \) and \( \sinh x \) satisfy the relation

\[
cosh^2 x - \sinh^2 x \equiv 1
\]

which is very similar to the trigonometric identity \( \cos^2 x + \sin^2 x \equiv 1 \). (In fact every trigonometric identity has an equivalent hyperbolic function identity.)

The hyperbolic functions are not introduced because they are a mathematical nicety. They arise naturally and sufficiently often to warrant sustained study. For example, the shape of a chain hanging under gravity is well described by \( \cosh \) and the deformation of uniform beams can be expressed in terms of \( \tanh \).

Prerequisites

Before starting this Section you should . . .

- have a good knowledge of the exponential function
- have knowledge of odd and even functions
- have familiarity with the definitions of \( \tan \), \( \sec \), \( \cosec \), \( \cot \) and of trigonometric identities

Learning Outcomes

On completion you should be able to . . .

- explain how hyperbolic functions are defined in terms of exponential functions
- obtain and use hyperbolic function identities
- manipulate expressions involving hyperbolic functions
1. Even and odd functions

Constructing even and odd functions

A given function \( f(x) \) can always be split into two parts, one of which is even and one of which is odd. To do this write \( f(x) \) as \( \frac{1}{2}[f(x) + f(-x)] \) and then simply add and subtract \( \frac{1}{2}f(-x) \) to this to give

\[
f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]
\]

The term \( \frac{1}{2}[f(x) + f(-x)] \) is \textbf{even} because when \( x \) is replaced by \( -x \) we have \( \frac{1}{2}[f(-x) + f(x)] \) which is the same as the original. However, the term \( \frac{1}{2}[f(x) - f(-x)] \) is \textbf{odd} since, on replacing \( x \) by \( -x \) we have \( \frac{1}{2}[f(-x) - f(x)] = -\frac{1}{2}[f(x) - f(-x)] \) which is the negative of the original.

**Example 2**

Separate \( x^3 + 2^x \) into odd and even parts.

**Solution**

\[
f(x) = x^3 + 2^x \\
f(-x) = (-x)^3 + 2^{-x} = -x^3 + 2^{-x}
\]

Even part:

\[
\frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}(x^3 + 2^x - x^3 + 2^{-x}) = \frac{1}{2}(2^x + 2^{-x})
\]

Odd part:

\[
\frac{1}{2}(f(x) - f(-x)) = \frac{1}{2}(x^3 + 2^x + x^3 - 2^{-x}) = \frac{1}{2}(2x^3 + 2^x - 2^{-x})
\]

**Task**

Separate the function \( x^2 - 3^x \) into odd and even parts.

First, define \( f(x) \) and find \( f(-x) \):

**Your solution**

\[
f(x) = \] \qquad f(-x) =
\]

**Answer**

\( f(x) = x^2 - 3^x, \quad f(-x) = x^2 - 3^{-x} \)
Now construct \( \frac{1}{2}[f(x) + f(-x)], \ \frac{1}{2}[f(x) - f(-x)]; \)

<table>
<thead>
<tr>
<th>Your solution</th>
<th>( \frac{1}{2}[f(x) + f(-x)] = )</th>
<th>( \frac{1}{2}[f(x) - f(-x)] = )</th>
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\[
\frac{1}{2}[f(x) + f(-x)] = \frac{1}{2}(x^2 - 3^x + x^2 - 3^{-x}) = x^2 - \frac{1}{2}(3^x + 3^{-x}). \text{ This is the even part of } f(x).
\]

\[
\frac{1}{2}[f(x) - f(-x)] = \frac{1}{2}(x^2 - 3^x - x^2 + 3^{-x}) = \frac{1}{2}(3^{-x} - 3^x). \text{ This is the odd part of } f(x).
\]

The odd and even parts of the exponential function

Using the approach outlined above we see that the even part of \( e^x \) is

\[
\frac{1}{2}(e^x + e^{-x})
\]

and the odd part of \( e^x \) is

\[
\frac{1}{2}(e^x - e^{-x})
\]

We give these new functions special names: \( \cosh x \) (pronounced ‘cosh’ \( x \)) and \( \sinh x \) (pronounced ‘shine’ \( x \)).

<table>
<thead>
<tr>
<th>Key Point 3</th>
<th>Hyperbolic Functions</th>
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</thead>
<tbody>
<tr>
<td>( \cosh x \equiv \frac{1}{2}(e^x + e^{-x}) )</td>
<td></td>
</tr>
<tr>
<td>( \sinh x \equiv \frac{1}{2}(e^x - e^{-x}) )</td>
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These two functions, when added and subtracted, give

\[
\cosh x + \sinh x \equiv e^x \quad \text{and} \quad \cosh x - \sinh x \equiv e^{-x}
\]

The graphs of \( \cosh x \) and \( \sinh x \) are shown in Figure 4.
Figure 4: sinh $x$ and cosh $x$

Note that cosh $x > 0$ for all values of $x$ and that sinh $x$ is zero only when $x = 0$.

2. Hyperbolic identities

The hyperbolic functions cosh $x$, sinh $x$ satisfy similar (but not exactly equivalent) identities to those satisfied by cos $x$, sin $x$. We note first some basic notation similar to that employed with trigonometric functions:

$cosh^n x$ means $(cosh x)^n$ \hspace{1cm} sinh$^n x$ means $(sinh x)^n$ \hspace{1cm} $n \neq -1$

In the special case that $n = -1$ we do not use $cosh^{-1} x$ and sinh$^{-1} x$ to mean $\frac{1}{cosh x}$ and $\frac{1}{sinh x}$ respectively. The notation $cosh^{-1} x$ and sinh$^{-1} x$ is reserved for the inverse functions of $cosh x$ and sinh $x$ respectively.

Task

Show that $\cosh^2 x - \sinh^2 x \equiv 1$ for all $x$.

(a) First, express $\cosh^2 x$ in terms of the exponential functions $e^x$, $e^{-x}$:

Your solution

$\cosh^2 x \equiv \left[ \frac{1}{2}(e^x + e^{-x}) \right]^2 \equiv$

Answer

$\frac{1}{4}(e^x + e^{-x})^2 \equiv \frac{1}{4}[(e^x)^2 + 2e^xe^{-x} + (e^{-x})^2] \equiv \frac{1}{4}[e^{2x} + 2e^xe^{-x} + e^{-2x}] \equiv \frac{1}{4}[e^{2x} + 2 + e^{-2x}]$
(b) Similarly, express $\sinh^2 x$ in terms of $e^x$ and $e^{-x}$:

Your solution

$$\sinh^2 x \equiv \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \equiv$$

Answer

$$\frac{1}{4}(e^x - e^{-x})^2 \equiv \frac{1}{4}[(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2] \equiv \frac{1}{4}[e^{2x} - 2e^x e^{-x} + e^{-2x}] \equiv \frac{1}{4}[e^{2x} - 2 + e^{-2x}]$$

(c) Finally determine $\cosh^2 x - \sinh^2 x$ using the results from (a) and (b):

Your solution

$$\cosh^2 x - \sinh^2 x \equiv$$

Answer

$$\cosh^2 x - \sinh^2 x \equiv \frac{1}{4}[e^{2x} + 2 + e^{-2x}] - \frac{1}{4}[e^{2x} - 2 + e^{-2x}] \equiv 1$$

As an alternative to the calculation in this Task we could, instead, use the relations

$$e^x \equiv \cosh x + \sinh x \quad e^{-x} \equiv \cosh x - \sinh x$$

and remembering the algebraic identity $(a + b)(a - b) \equiv a^2 - b^2$, we see that

$$(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv e^x e^{-x} \equiv 1 \quad \text{that is} \quad \cosh^2 x - \sinh^2 x \equiv 1$$

Key Point 4

The fundamental identity relating hyperbolic functions is:

$$\cosh^2 x - \sinh^2 x \equiv 1$$

This is the hyperbolic function equivalent of the trigonometric identity: $\cos^2 x + \sin^2 x \equiv 1$
Show that \( \cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y \).

First, express \( \cosh x \cosh y \) in terms of exponentials:

**Your solution**
\[
\cosh x \cosh y \equiv \left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^y + e^{-y}}{2} \right)
\]

**Answer**
\[
\left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^y + e^{-y}}{2} \right) \equiv \frac{1}{4} \left[ e^x e^y + e^{-x} e^y + e^x e^{-y} + e^{-x} e^{-y} \right] \equiv \frac{1}{4} (e^{x+y} + e^{-x+y} + e^{x-y} + e^{-x-y})
\]

Now express \( \sinh x \sinh y \) in terms of exponentials:

**Your solution**
\[
\left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^y - e^{-y}}{2} \right)
\]

**Answer**
\[
\left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^y - e^{-y}}{2} \right) \equiv \frac{1}{4} (e^{x+y} - e^{-x+y} - e^{x-y} + e^{-x-y})
\]

Now express \( \cosh x \cosh y + \sinh x \sinh y \) in terms of a hyperbolic function:

**Your solution**
\[
\cosh x \cosh y + \sinh x \sinh y
\]

**Answer**
\[
\cosh x \cosh y + \sinh x \sinh y \equiv \frac{1}{2} (e^{x+y} + e^{-(x+y)}) \text{ which we recognise as } \cosh(x + y)
\]
Other hyperbolic function identities can be found in a similar way. The most commonly used are listed in the following Key Point.

### Key Point 5

**Hyperbolic Identities**

- \( \cosh^2 x - \sinh^2 x \equiv 1 \)
- \( \cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y \)
- \( \sinh(x + y) \equiv \sinh x \cosh y + \cosh x \sinh y \)
- \( \sinh 2x \equiv 2 \sinh x \cosh y \)
- \( \cosh 2x \equiv \cosh^2 x + \sinh^2 x \) or \( \cosh 2x \equiv 2 \cosh^2 x - 1 \) or \( \cosh 2x \equiv 1 + 2 \sinh^2 x \)

### 3. Related hyperbolic functions

Given the trigonometric functions \( \cos x, \sin x \) related functions can be defined; \( \tan x, \sec x, \cosec x \) through the relations:

- \( \tan x \equiv \frac{\sin x}{\cos x} \)
- \( \sec x \equiv \frac{1}{\cos x} \)
- \( \cosec x \equiv \frac{1}{\sin x} \)
- \( \cot x \equiv \frac{\cos x}{\sin x} \)

In an analogous way, given \( \cosh x \) and \( \sinh x \) we can introduce hyperbolic functions \( \tanh x, \sech x, \cosech x \) and \( \coth x \). These functions are defined in the following Key Point:

### Key Point 6

**Further Hyperbolic Functions**

- \( \tanh x \equiv \frac{\sinh x}{\cosh x} \)
- \( \sech x \equiv \frac{1}{\cosh x} \)
- \( \cosech x \equiv \frac{1}{\sinh x} \)
- \( \coth x \equiv \frac{\cosh x}{\sinh x} \)
Show that \( 1 - \tanh^2 x \equiv \sech^2 x \)

Use the identity \( \cosh^2 x - \sinh^2 x \equiv 1 \):

**Your solution**

**Answer**

Dividing both sides by \( \cosh^2 x \) gives

\[
1 - \frac{\sinh^2 x}{\cosh^2 x} \equiv 1 \frac{1}{\cosh^2 x} \quad \text{implying (see Key Point 6)} \quad 1 - \tanh^2 x \equiv \sech^2 x
\]

**Exercises**

1. Express
   
   (a) \( 2 \sinh x + 3 \cosh x \) in terms of \( e^x \) and \( e^{-x} \).
   (b) \( 2 \sinh 4x - 7 \cosh 4x \) in terms of \( e^{4x} \) and \( e^{-4x} \).

2. Express
   
   (a) \( 2e^x - e^{-x} \) in terms of \( \sinh x \) and \( \cosh x \).
   (b) \( \frac{7e^x}{(e^x - e^{-x})} \) in terms of \( \sinh x \) and \( \cosh x \), and then in terms of \( \coth x \).
   (c) \( 4e^{-3x} - 3e^{3x} \) in terms of \( \sinh 3x \) and \( \cosh 3x \).

3. Using only the \( \cosh \) and \( \sinh \) keys on your calculator (or \( e^x \) key) find the values of
   
   (a) \( \tanh 0.35 \), \quad (b) \( \cosech 2 \), \quad (c) \( \sech 0.6 \).

**Answers**

1. (a) \( \frac{5}{2}e^x - \frac{1}{2}e^{-x} \) \quad (b) \( \frac{5}{2}e^{4x} - \frac{9}{2}e^{-4x} \)

2. (a) \( \cosh x + 3 \sinh x \), \quad (b) \( \frac{7(\cosh x + \sinh x)}{2 \sinh x} \), \quad \( \frac{7}{2}(\coth x + 1) \) \quad (c) \( \cosh 3x - 7 \sinh 3x \)

3. (a) 0.3364, \quad (b) 0.2757 \quad (c) 0.8436