Introduction

In this Section we introduce Cauchy’s theorem which allows us to simplify the calculation of certain contour integrals. A second result, known as Cauchy’s integral formula, allows us to evaluate some integrals of the form \[ \oint_{C} \frac{f(z)}{z - z_0} \, dz \] where \( z_0 \) lies inside \( C \).

Prerequisites

Before starting this Section you should . . .

- be familiar with the basic ideas of functions of a complex variable
- be familiar with line integrals

Learning Outcomes

On completion you should be able to . . .

- state and use Cauchy’s theorem
- state and use Cauchy’s integral formula
1. Cauchy’s theorem

Simply-connected regions

A region is said to be simply-connected if any closed curve in that region can be shrunk to a point without any part of it leaving a region. The interior of a square or a circle are examples of simply connected regions. In Figure 11 (a) and (b) the shaded grey area is the region and a typical closed curve is shown inside the region. In Figure 11 (c) the region contains a hole (the white area inside). The shaded region between the two circles is not simply-connected; curve $C_1$ can shrink to a point but curve $C_2$ cannot shrink to a point without leaving the region, due to the hole inside it.

![Figure 11](image)

**Figure 11**

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**Key Point 2**

*Cauchy’s Theorem*

The theorem states that if $f(z)$ is analytic everywhere within a simply-connected region then:

$$\oint_C f(z) \, dz = 0$$

for every simple closed path $C$ lying in the region.

This is perhaps the most important theorem in the area of complex analysis.

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As a straightforward example note that $\oint_C z^2 \, dz = 0$, where $C$ is the unit circle, since $z^2$ is analytic everywhere (see Section 261). Indeed $\oint_C z^2 \, dz = 0$ for any simple contour: it need not be circular.

Consider the contour shown in Figure 12 and assume $f(z)$ is analytic everywhere on and inside the
contour $C$.

Figure 12

Then by analogy with real line integrals

$$\int_{AEB} f(z) \, dz + \int_{BDA} f(z) \, dz = \oint_C f(z) \, dz = 0$$

by Cauchy’s theorem.

Therefore

$$\int_{AEB} f(z) \, dz = -\int_{BDA} f(z) \, dz = \int_{ADB} f(z) \, dz$$

(since reversing the direction of integration reverses the sign of the integral).

This implies that we may choose any path between $A$ and $B$ and the integral will have the same value providing $f(z)$ is analytic in the region concerned.

Integrals of analytic functions only depend on the positions of the points $A$ and $B$, not on the path connecting them. This explains the ‘coincidences’ referred to previously in Section 26.4.

**Task**

Using ‘simple’ integration evaluate $\int_i^{1+2i} \cos z \, dz$, and explain why this is valid.

**Your solution**

**Answer**

$$\int_i^{1+2i} \cos z \, dz = \left[ \sin z \right]_i^{1+2i} = \sin(1+2i) - \sin i.$$

This way of determining the integral is legitimate because $\cos z$ is analytic (everywhere).
We now investigate what occurs when the closed path of integration does not necessarily lie within a simply-connected region. Consider the situation described in Figure 13.

Let $f(z)$ be analytic in the region bounded by the closed curves $C_1$ and $C_2$. The region is cut by the line segment joining $A$ and $B$.

Consider now the closed curve $AEABFBA$ travelling in the direction indicated by the arrows. No line can cross the cut $AB$ and be regarded as remaining in the region. Because of the cut the shaded region is simply connected. Cauchy’s theorem therefore applies (see Key Point 2).

Therefore
\[ \oint_{AEABFBA} f(z) \, dz = 0 \] since $f(z)$ is analytic within and on the curve $AEABFBA$.

Note that
\[ \int_{AB} f(z) \, dz = -\int_{BA} f(z) \, dz, \] being a simple change of direction.

Also, we can divide the closed curve into smaller sections:
\[
\oint_{AEABFBA} f(z) \, dz = \int_{AEA} f(z) \, dz + \int_{AB} f(z) \, dz + \int_{BFB} f(z) \, dz + \int_{BA} f(z) \, dz
\]
\[= \int_{AEA} f(z) \, dz + \int_{BFB} f(z) \, dz = 0.\]

i.e.
\[\oint_{C_1} f(z) \, dz - \oint_{C_2} f(z) \, dz = 0\]
(since we assume that closed paths are travelled anticlockwise).

Therefore
\[ \oint_{C_1} f(z) \, dz = \oint_{C_2} f(z) \, dz.\]

This allows us to evaluate $\oint_{C_1} f(z) \, dz$ by replacing $C_1$ by any curve $C_2$ such that the region between them contains no singularities (see Section 261) of $f(z)$. Often we choose a circle for $C_2$. 
Example 12

Determine \( \oint_C \frac{6}{z(z-3)} \, dz \) where \( C \) is the curve \( |z-3| = 5 \) shown in Figure 14.

Solution

We observe that \( f(z) = \frac{6}{z(z-3)} \) is analytic everywhere except at \( z = 0 \) and \( z = 3 \).

Let \( C_1 \) be the circle of unit radius centred at \( z = 3 \) and \( C_2 \) be the unit circle centered at the origin.

By analogy with the previous example we state that

\[
\oint_C \frac{6}{z(z-3)} \, dz = \oint_{C_1} \frac{6}{z(z-3)} \, dz + \oint_{C_2} \frac{6}{z(z-3)} \, dz.
\]

(To show this you would need two cuts: from \( C \) to \( C_1 \) and from \( C \) to \( C_2 \).)

The remaining parts of this problem are presented as two Tasks.

Task

Expand \( \frac{6}{z(z-3)} \) into partial functions.

Your solution

Answer

Let \( \frac{6}{z(z-3)} \equiv \frac{A}{z} + \frac{B}{z-3} \equiv \frac{A(z-3) + Bz}{z(z-3)} \). Then \( A(z-3) + Bz \equiv 6 \).

If \( z = 0 \) \( A(-3) = 6 \) \( \therefore A = -2 \). If \( z = 3 \) \( B \times 3 = 6 \) \( \therefore B = 2 \).

\[
\therefore \frac{6}{z(z-3)} \equiv -\frac{2}{z} + \frac{2}{z-3}.
\]
Thus:
\[ \oint_C z(z - 3) \, dz = \oint_{C_1} \frac{2}{z - 3} \, dz - \oint_{C_1} \frac{2}{z} \, dz + \oint_{C_2} \frac{2}{z - 3} \, dz - \oint_{C_2} \frac{2}{z} \, dz = I_1 - I_2 + I_3 - I_4. \]

**Task**
Find the values of $I_1, I_2, I_3, I_4$, using Key Point 1 (page 35):

(a) Find the value of $I_1$:

**Your solution**

**Answer**
Using Key Point 1 we find that $I_1 = 2 \times 2\pi i = 4\pi i$.

(b) Find the value of $I_2$:

**Your solution**

**Answer**
The function $\frac{1}{z}$ is analytic inside and on $C_1$ so that $I_2 = 0$.

(c) Find the value of $I_3$:

**Your solution**

**Answer**
The function $\frac{1}{z - 3}$ is analytic inside and on $C_2$ so $I_3 = 0$.

(d) Find the value of $I_4$:

**Your solution**

**Answer**
$I_4 = 4\pi i$ again using Key Point 1.

(e) Finally, calculate $I = I_1 - I_2 + I_3 - I_4$:

**Your solution**
\[ \oint_C \frac{6 \, dz}{z(z-3)} = 4\pi i - 0 + 0 - 4\pi i = 0. \]

**Exercises**

1. Evaluate \[ \int_{1+i}^{2+3i} \sin z \, dz. \]
2. Determine \[ \oint_C \frac{4}{z(z-2)} \, dz \] where \( C \) is the contour \(|z-2|=4\).

**Answers**

1. \[ \int_{1+i}^{2+3i} \sin z \, dz = \left[ -\cos z \right]_{1+i}^{2+3i} = \cos(1+i) - \cos(2+3i) \] since \( \sin z \) is analytic everywhere.

2. \[ f(z) = \frac{4}{z(z-2)} \] is analytic everywhere except at \( z=0 \) and \( z=2 \).

Call \[ I = \oint_C \frac{4}{z(z-2)} \, dz = \oint_{C_1} \frac{4}{z(z-2)} \, dz + \oint_{C_2} \frac{4}{z(z-2)} \, dz. \]

Now \[ \frac{4}{z(z-2)} \equiv -\frac{2}{z} + \frac{2}{z-2} \] so that

\[ I = \oint_{C_1} \frac{2}{z-2} \, dz - \oint_{C_1} \frac{2}{z} \, dz + \oint_{C_2} \frac{2}{z} \, dz - \oint_{C_2} \frac{2}{z-2} \, dz = I_1 + I_2 + I_3 + I_4. \]

\( I_2 \) and \( I_3 \) are zero because of analyticity.

\( I_1 = 2 \times 2\pi i = 4\pi i \), by Key Point 1 and \( I_4 = -4\pi i \) likewise.

Hence \( I = 4\pi i + 0 + 0 - 4\pi i = 0. \)
2. Cauchy’s integral formula

This is a generalization of the result in Key Point 2:

Key Point 3
Cauchy’s Integral Formula

If \( f(z) \) is analytic inside and on the boundary \( C \) of a simply-connected region then for any point \( z_0 \) inside \( C \),

\[
\oint_C \frac{f(z)}{z-z_0} \, dz = 2\pi i \, f(z_0).
\]

Example 13
Evaluate \( \oint_C \frac{z}{z^2 + 1} \, dz \) where \( C \) is the path shown in Figure 15:

\( C_1 \): \(|z - i| = \frac{1}{2}\)

Solution
We note that \( z^2 + 1 \equiv (z + i)(z - i) \).

Let \( \frac{z}{z^2 + 1} = \frac{z}{(z + i)(z - i)} = \frac{z/(z + i)}{z - i} \).

The numerator \( z/(z + i) \) is analytic inside and on the path \( C_1 \) so putting \( z_0 = i \) in the Cauchy integral formula (Key Point 3)

\[
\oint_{C_1} \frac{z}{z^2 + 1} \, dz = 2\pi i \left[ \frac{i}{i + i} \right] = 2\pi i \cdot \frac{1}{2} = \pi i.
\]
Evaluate \( \oint_C \frac{z}{z^2 + 1} \, dz \) where \( C \) is the path (refer to the diagram).

(a) \( C_2 : \quad |z + i| = \frac{1}{2} \) \quad (b) \( C_3 : \quad |z| = 2 \).

(a) Use the Cauchy integral formula to find an expression for \( \oint_{C_2} \frac{z}{z^2 + 1} \, dz \):

Your solution

Answer

\[
\frac{z}{z^2 + 1} = \frac{z}{(z - i)}.
\]

The numerator is analytic inside and on the path \( C_2 \) so putting \( z_0 = -i \) in the Cauchy integral formula gives

\[
\oint_{C_2} \frac{z}{z^2 + 1} \, dz = 2\pi i \left[ \frac{-i}{-2i} \right] = \pi i.
\]

(b) Now find \( \oint_{C_3} \frac{z}{z^2 + 1} \, dz \):

Your solution

Answer

By analogy with the previous part,

\[
\oint_{C_3} \frac{z}{z^2 + 1} \, dz = \oint_{C_1} \frac{z}{z^2 + 1} \, dz + \oint_{C_2} \frac{z}{z^2 + 1} \, dz = \pi i + \pi i = 2\pi i.
\]
The derivative of an analytic function

If \( f(z) \) is analytic in a simply-connected region then at any interior point of the region, \( z_0 \) say, the derivatives of \( f(z) \) of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!). The derivatives at the point \( z_0 \) are given by Cauchy's integral formula for derivatives:

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz
\]

where \( C \) is any simple closed curve, in the region, which encloses \( z_0 \).

Note the case \( n = 1 \):

\[
f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} \, dz.
\]

**Example 14**

Evaluate the contour integral

\[\oint_C \frac{z^3}{(z-1)^2} \, dz\]

where \( C \) is a contour which encloses the point \( z = 1 \).

**Solution**

Since \( f(z) = \frac{z^3}{(z-1)^2} \) has a pole of order 2 at \( z = 1 \) then

\[\oint_C f(z) \, dz = \oint_{C'} \frac{z^3}{(z-1)^2} \, dz\]

where \( C' \) is a circle centered at \( z = 1 \).

If \( g(z) = z^3 \) then

\[\oint_C f(z) \, dz = \oint_{C'} \frac{g(z)}{(z-1)^2} \, dz\]

Since \( g(z) \) is analytic within and on the circle \( C' \) we use Cauchy's integral formula for derivatives to show that

\[\oint_C \frac{z^3}{(z-1)^2} \, dz = 2\pi i \times \frac{1}{1!} [g'(z)]_{z=1} = 2\pi i \left[3z^2\right]_{z=1} = 6\pi i.\]
Exercise

Evaluate \( \oint_C \frac{z}{z^2 + 9} \, dz \) where \( C \) is the path:

(a) \( C_1 : \quad |z - 3i| = 1 \)
(b) \( C_2 : \quad |z + 3i| = 1 \)
(c) \( C_3 : \quad |z| = 6 \).

Answers

(a) We will use the fact that \( \frac{z}{z^2 + 9} = \frac{z}{(z + 3i)(z - 3i)} = \frac{z/(z + 3i)}{z - 3i} \)

The numerator \( \frac{z}{z + 3i} \) is analytic inside and on the path \( C_1 \) so putting \( z_0 = 3i \) in Cauchy’s integral formula:

\[
\oint_{C_1} \frac{z}{z^2 + 9} \, dz = 2\pi i \left[ \frac{3i}{3i + 3i} \right] = 2\pi i \times \frac{1}{2} = \pi i.
\]

(b) Here \( \frac{z/(z - 3i)}{z + 3i} \)

The numerator is analytic inside and on the path \( C_2 \) so putting \( z = -3i \) in Cauchy’s integral formula:

\[
\oint_{C_2} \frac{z}{z^2 + 9} \, dz = 2\pi i \left[ \frac{-3i}{-3i - 3i} \right] = \pi i.
\]

(c) The integral is the sum of the two previous integrals and has value \( 2\pi i \).