Introduction

In this Section we address the following problem:

Can we find a Fourier series expansion of a function defined over a finite interval?

Of course we recognise that such a function could not be periodic (as periodicity demands an infinite interval). The answer to this question is yes but we must first convert the given non-periodic function into a periodic function. There are many ways of doing this. We shall concentrate on the most useful extension to produce a so-called half-range Fourier series.

Prerequisites

Before starting this Section you should . . .

- know how to obtain a Fourier series
- be familiar with odd and even functions and their properties
- have knowledge of integration by parts

Learning Outcomes

On completion you should be able to . . .

- choose to expand a non-periodic function either as a series of sines or as a series of cosines
1. Half-range Fourier series

So far we have shown how to represent given periodic functions by Fourier series. We now consider a slight variation on this theme which will be useful in HELM 25 on solving Partial Differential Equations.

Suppose that instead of specifying a periodic function we begin with a function $f(t)$ defined only over a **limited range of values** of $t$, say $0 < t < \pi$. Suppose further that we wish to represent this function, over $0 < t < \pi$, by a Fourier series. (This situation may seem a little artificial at this point, but this is precisely the situation that will arise in solving differential equations.)

To be specific, suppose we define $f(t) = t^2$, $0 < t < \pi$

We shall consider the interval $0 < t < \pi$ to be half a period of a $2\pi$ periodic function. We must therefore define $f(t)$ for $-\pi < t < 0$ to complete the specification.

**Task**

Complete the definition of the above function $f(t) = t^2$, $0 < t < \pi$
by defining it over $-\pi < t < 0$ such that the resulting functions will have a Fourier series containing

(a) only cosine terms, (b) only sine terms, (c) both cosine and sine terms.

**Your solution**
(a) We must complete the definition so as to have an even periodic function:
\[ f(t) = t^2, \quad -\pi < t < 0 \]

(b) We must complete the definition so as to have an odd periodic function:
\[ f(t) = -t^2, \quad -\pi < t < 0 \]

(c) We may define \( f(t) \) in any way we please (other than (a) and (b) above). For example we might define \( f(t) = 0 \) over \(-\pi < t < 0\):

The point is that all three periodic functions \( f_1(t), f_2(t), f_3(t) \) will give rise to a different Fourier series but all will represent the function \( f(t) = t^2 \) over \( 0 < t < \pi \). Fourier series obtained by extending functions in this sort of way are often referred to as half-range series.

Normally, in applications, we require either a Fourier Cosine series (so we would complete a definition as in (i) above to obtain an even periodic function) or a Fourier Sine series (for which, as in (ii) above, we need an odd periodic function.)

The above considerations apply equally well for a function defined over any interval.
Example 3

Obtain the half range Fourier Sine series to represent \( f(t) = t^2 \quad 0 < t < 3 \).

Solution

We first extend \( f(t) \) as an odd periodic function \( F(t) \) of period 6: \( f(t) = -t^2, \quad -3 < t < 0 \)

\[
\begin{align*}
\text{Figure 22}
\end{align*}
\]

We now evaluate the Fourier series of \( F(t) \) by standard techniques but take advantage of the symmetry and put \( a_n = 0, \quad n = 0, 1, 2, \ldots \).

Using the results for the Fourier Sine coefficients for period \( T \) from HELM 23.2 subsection 5, \( b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} F(t) \sin \left(\frac{2n\pi t}{T}\right) dt, \)

we put \( T = 6 \) and, since the integrand is even (a product of 2 odd functions), we can write

\[
b_n = \frac{2}{3} \int_0^3 F(t) \sin \left(\frac{2n\pi t}{6}\right) dt = \frac{2}{3} \int_0^3 t^2 \sin \left(\frac{n\pi t}{3}\right) dt.
\]

(Note that we always integrate over the originally defined range, in this case \( 0 < t < 3 \).)

We now have to integrate by parts (twice!)

\[
b_n = \frac{2}{3} \left\{ \left[ \frac{-3t^2}{n\pi} \cos \left(\frac{n\pi t}{3}\right) \right]_0^3 + 2 \left( \frac{3}{n\pi} \right) \int_0^3 t \cos \left(\frac{n\pi t}{3}\right) dt \right\}
\]

\[
= \frac{2}{3} \left\{ \left[ \frac{-27}{n\pi} \cos n\pi + \frac{6}{n\pi} \left( \frac{3}{n\pi} t \sin \frac{n\pi t}{3} \right) \right]_0^3 - \left( \frac{6}{n\pi} \right) \left( \frac{3}{n\pi} \right) \int_0^3 \sin \left(\frac{n\pi t}{3}\right) dt \right\}
\]

\[
= \frac{2}{3} \left\{ \left[ \frac{-27}{n\pi} \cos n\pi - \frac{18}{n^2\pi^2} \left[ \frac{3}{n\pi} \cos \left(\frac{n\pi t}{3}\right) \right]^3 \right]_0^3 - \left( \frac{6}{n\pi} \right) \left( \frac{3}{n\pi} \right) \int_0^3 \sin \left(\frac{n\pi t}{3}\right) dt \right\}
\]

\[
= \frac{2}{3} \left\{ \frac{-18}{n\pi} \quad n = 2, 4, 6, \ldots \right\}
\]

\[
= \left\{ \frac{18}{n\pi} - \frac{72}{n^3\pi^3} \quad n = 1, 3, 5, \ldots \right\}
\]

So the required Fourier Sine series is

\[
F(t) = 18 \left( \frac{1}{\pi} - \frac{4}{\pi^3} \right) \sin \left(\frac{\pi t}{3}\right) - 18 \left( \frac{2}{2\pi} \sin \left(\frac{2\pi t}{3}\right) + 18 \left( \frac{1}{3\pi} - \frac{4}{27\pi^3} \right) \sin(\pi t) - \ldots
\]
Obtain a half-range Fourier Cosine series to represent the function

\[ f(t) = 4 - t \quad 0 < t < 4. \]

First complete the definition to obtain an even periodic function \( F(t) \) of period 8. Sketch \( F(t) \):

**Your solution**

![Sketch of \( F(t) \)]

**Answer**

Now formulate the integral from which the Fourier coefficients \( a_n \) can be calculated:

**Your solution**

![Integral formula]

**Answer**

We have with \( T = 8 \)

\[
a_n = \frac{2}{8} \int_{-4}^{4} F(t) \cos \left( \frac{2n\pi t}{8} \right) \, dt
\]

Utilising the fact that the integrand here is even we get

\[
a_n = \frac{1}{2} \int_{0}^{4} (4 - t) \cos \left( \frac{n\pi t}{4} \right) \, dt
\]
Now integrate by parts to obtain \( a_n \) and also obtain \( a_0 \):

**Your solution**

**Answer**

Using integration by parts we obtain for \( n = 1, 2, 3, \ldots \)

\[
a_n = \frac{1}{2} \left\{ \left[ (4-t) \frac{4}{n\pi} \sin \left( \frac{n\pi t}{4} \right) \right]_0^4 + \frac{4}{n\pi} \int_0^4 \sin \left( \frac{n\pi t}{4} \right) \, dt \right\}
\]

\[
= \frac{1}{2} \left( \frac{4}{n\pi} \right) \left( \frac{4}{n\pi} \right) \left[ -\cos \left( \frac{n\pi t}{4} \right) \right]_0^4 = \frac{8}{n^2\pi^2} \left[ -\cos(n\pi) + 1 \right]
\]

i.e. \( a_n = \begin{cases} 0 & n = 2, 4, 6, \ldots \\ \frac{16}{n^2\pi^2} & n = 1, 3, 5, \ldots \end{cases} \)

Also \( a_0 = \frac{1}{2} \int_0^4 (4-t) \, dt = 4 \). So the constant term is \( \frac{a_0}{2} = 2 \).

Now write down the required Fourier series:

**Your solution**

**Answer**

We get

\[
2 + \frac{16}{\pi^2} \left\{ \cos \left( \frac{\pi t}{4} \right) + \frac{1}{9} \cos \left( \frac{3\pi t}{4} \right) + \frac{1}{25} \cos \left( \frac{5\pi t}{4} \right) + \ldots \right\}
\]
Note that the form of the Fourier series (a constant of 2 together with odd harmonic cosine terms) could be predicted if, in the sketch of $F(t)$, we imagine raising the $t$-axis by 2 units i.e. writing

$$F(t) = 2 + G(t)$$

![Figure 23](image)

Clearly $G(t)$ possesses half-period symmetry

$$G(t + 4) = -G(t)$$

and hence its Fourier series must contain only odd harmonics.

**Exercises**

Obtain the half-range Fourier series specified for each of the following functions:

1. $f(t) = 1 \quad 0 \leq t \leq \pi$ (sine series)
2. $f(t) = t \quad 0 \leq t \leq 1$ (sine series)
3. (a) $f(t) = e^{2t} \quad 0 \leq t \leq 1$ (cosine series)
   (b) $f(t) = e^{2t} \quad 0 \leq t \leq \pi$ (sine series)
4. (a) $f(t) = \sin t \quad 0 \leq t \leq \pi$ (cosine series)
   (b) $f(t) = \sin t \quad 0 \leq t \leq \pi$ (sine series)

**Answers**

1. $\frac{4}{\pi} \left\{ \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \cdots \right\}$
2. $\frac{2}{\pi} \{ \sin \pi t - \frac{1}{2} \sin 2\pi t + \frac{1}{3} \sin 3\pi t - \cdots \}$
3. (a) $\frac{e^2 - 1}{2} + \sum_{n=1}^{\infty} \frac{4}{4 + n^2 \pi^2} \{ e^n \cos(n\pi) - 1 \} \cos n\pi t$
   (b) $\sum_{n=1}^{\infty} \frac{2n\pi}{4 + n^2 \pi^2} \{ 1 - e^n \cos(n\pi) \} \sin n\pi t$
4. (a) $\frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{1}{\pi} \left\{ \frac{1}{1 - n} (1 - \cos(1 - n)\pi) + \frac{1}{1 + n} (1 - \cos(1 + n)\pi) \right\} \cos nt$
   (b) $\sin t$ itself (!)