The Binomial Series

16.3

Introduction

In this Section we examine an important example of an infinite series, the binomial series:

\[ 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots \]

We show that this series is only convergent if \(|x| < 1\) and that in this case the series sums to the value \((1 + x)^p\). As a special case of the binomial series we consider the situation when \(p\) is a positive integer \(n\). In this case the infinite series reduces to a finite series and we obtain, by replacing \(x\) with \(\frac{b}{a}\), the binomial theorem:

\[(b + a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{2!}b^{n-2}a^2 + \ldots + a^n.\]

Finally, we use the binomial series to obtain various polynomial expressions for \((1 + x)^p\) when \(x\) is ‘small’.

Prerequisites

Before starting this Section you should . . .

- understand the factorial notation
- have knowledge of the ratio test for convergence of infinite series.
- understand the use of inequalities

Learning Outcomes

On completion you should be able to . . .

- recognise and use the binomial series
- state and use the binomial theorem
- use the binomial series to obtain numerical approximations
1. The binomial series

A very important infinite series which occurs often in applications and in algebra has the form:

\[ 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots \]

in which \( p \) is a given number and \( x \) is a variable. By using the ratio test it can be shown that this series converges, irrespective of the value of \( p \), as long as \( |x| < 1 \). In fact, as we shall see in Section 16.5 the given series converges to the value \((1 + x)^p\) as long as \( |x| < 1 \).

**Key Point 9**
The Binomial Series

\[
(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots \quad |x| < 1
\]

The binomial theorem can be obtained directly from the binomial series if \( p \) is chosen to be a positive integer (here we need not demand that \( |x| < 1 \) as the series is now finite and so is always convergent irrespective of the value of \( x \)). For example, with \( p = 2 \) we obtain

\[
(1 + x)^2 = 1 + 2x + \frac{2(1)}{2}x^2 + 0 + 0 + \cdots
\]

\[= 1 + 2x + x^2 \quad \text{as is well known.}\]

With \( p = 3 \) we get

\[
(1 + x)^3 = 1 + 3x + \frac{3(2)}{2}x^2 + \frac{3(2)(1)}{3!}x^3 + 0 + 0 + \cdots
\]

\[= 1 + 3x + 3x^2 + x^3\]

Generally if \( p = n \) (a positive integer) then

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n
\]

which is a form of the binomial theorem. If \( x \) is replaced by \( \frac{b}{a} \) then

\[
\left(1 + \frac{b}{a}\right)^n = 1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \cdots + \left(\frac{b}{a}\right)^n
\]

Now multiplying both sides by \( a^n \) we have the following Key Point:
Key Point 10
The Binomial Theorem

If \( n \) is a positive integer then the expansion of \((a + b)\) raised to the power \( n \) is given by:

\[
(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + b^n
\]

This is known as the binomial theorem.

Task
Use the binomial theorem to obtain  
(a) \((1 + x)^7\)  
(b) \((a + b)^4\)

(a) Here \( n = 7 \):

Your solution

\((1 + x)^7 = \)

Answer

\((1 + x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7\)

(b) Here \( n = 4 \):

Your solution

\((a + b)^4 = \)

Answer

\((a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\).

Task
Given that \( x \) is so small that powers of \( x^3 \) and above may be ignored in comparison to lower order terms, find a quadratic approximation of \((1 - x)^{\frac{1}{2}}\) and check for accuracy your approximation for \( x = 0.1 \).

First expand \((1 - x)^{\frac{1}{2}}\) using the binomial series with \( p = \frac{1}{2} \) and with \( x \) replaced by \((-x)\):

Your solution

\((1 - x)^{\frac{1}{2}} = \)
\[
(1 - x)^\frac{1}{2} = 1 - \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right)x^2 - \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^3 + \cdots 
\]

Now obtain the quadratic approximation:

**Your solution**

\[
(1 - x)^\frac{1}{2} \simeq 
\]

**Answer**

\[
(1 - x)^\frac{1}{2} \simeq 1 - \frac{1}{2}x - \frac{1}{8}x^2 
\]

Now check on the validity of the approximation by choosing \(x = 0.1\):

**Your solution**

On the left-hand side we have

\[
(0.9)^\frac{1}{2} = 0.94868 \text{ to 5 d.p. obtained by calculator}
\]

whereas, using the quadratic expansion:

\[
(0.9)^\frac{1}{2} \approx 1 - \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 = 1 - 0.05 - (0.00125) = 0.94875.
\]

so the error is only 0.00007.

What we have done in this last Task is to replace (or approximate) the function \((1 - x)^\frac{1}{2}\) by the simpler (polynomial) function \(1 - \frac{1}{2}x - \frac{1}{8}x^2\) which is reasonable provided \(x\) is very small. This approximation is well illustrated geometrically by drawing the curves \(y = (1 - x)^\frac{1}{2}\) and \(y = 1 - \frac{1}{2}x - \frac{1}{8}x^2\). The two curves coincide when \(x\) is ‘small’. See Figure 2:

![Figure 2](image_url)
Obtain a cubic approximation of $\frac{1}{(2 + x)}$. Check your approximation for accuracy using appropriate values of $x$.

First write the term $\frac{1}{(2 + x)}$ in a form suitable for the binomial series (refer to Key Point 9):

**Your solution**

$$\frac{1}{(2 + x)} =$$

**Answer**

$$\frac{1}{2 + x} = \frac{1}{2 \left(1 + \frac{x}{2}\right)} = \frac{1}{2} \left(1 + \frac{x}{2}\right)^{-1}$$

Now expand using the binomial series with $p = -1$ and $\frac{x}{2}$ instead of $x$, to include terms up to $x^3$:

**Your solution**

$$\frac{1}{2} \left(1 + \frac{x}{2}\right)^{-1} =$$

**Answer**

$$\frac{1}{2} \left(1 + \frac{x}{2}\right)^{-1} = \frac{1}{2} \left\{ 1 + (-1) \frac{x}{2} + \frac{(-1)(-2)}{2!} \left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!} \left(\frac{x}{2}\right)^3 \right\}$$

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$$

State the range of $x$ for which the binomial series of $\left(1 + \frac{x}{2}\right)^{-1}$ is valid:

**Your solution**

The series is valid if

**Answer**

valid as long as $\left|\frac{x}{2}\right| < 1$ i.e. $|x| < 2$ or $-2 < x < 2$
Choose $x = 0.1$ to check the accuracy of your approximation:

**Your solution**

\[
\frac{1}{2} \left( 1 + \frac{0.1}{2} \right)^{-1} = \frac{1}{2} - \frac{0.1}{4} + \frac{0.01}{8} - \frac{0.001}{16} =
\]

**Answer**

\[
\frac{1}{2} \left( 1 + \frac{0.1}{2} \right)^{-1} = 0.47619 \text{ to 5 d.p.}
\]

\[
\frac{1}{2} - \frac{0.1}{4} + \frac{0.01}{8} - \frac{0.001}{16} = 0.4761875.
\]

Figure 3 below illustrates the close correspondence (when $x$ is ‘small’) between the curves $y = \frac{1}{2}(1 + \frac{x}{2})^{-1}$ and $y = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$.

![Figure 3](image-url)

**Exercises**

1. Determine the expansion of each of the following
   
   (a) $(a + b)^3$, (b) $(1 - x)^5$, (c) $(1 + x^2)^{-1}$, (d) $(1 - x)^{1/3}$.

2. Obtain a cubic approximation (valid if $x$ is small) of the function $(1 + 2x)^{3/2}$.

**Answers**

1. (a) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
   (b) $(1 - x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$
   (c) $(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$
   (d) $(1 - x)^{1/3} = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + \cdots$

2. $(1 + 2x)^{3/2} = 1 + 3x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \cdots$

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