Parametric Differentiation

Introduction

Sometimes the equation of a curve is not be given in Cartesian form \( y = f(x) \) but in parametric form: \( x = h(t), \ y = g(t) \). In this Section we see how to calculate the derivative \( \frac{dy}{dx} \) from a knowledge of the so-called parametric derivatives \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \). We then extend this to the determination of the second derivative \( \frac{d^2 y}{dx^2} \).

Parametric functions arise often in particle dynamics in which the parameter \( t \) represents the time and \( (x(t), y(t)) \) then represents the position of a particle as it varies with time.

Prerequisites

Before starting this Section you should . . .
- be able to differentiate standard functions
- be able to plot a curve given in parametric form

Learning Outcomes

On completion you should be able to . . .
- find first and second derivatives when the equation of a curve is given in parametric form
1. Parametric differentiation

In this subsection we consider the parametric approach to describing a curve:

\[ \begin{align*}
  x &= h(t) \\
  y &= g(t) \\
  t_0 &\leq t \leq t_1
\end{align*} \]

parametric equations \hspace{1cm} parametric range

As various values of \( t \) are chosen within the parameter range the corresponding values of \( x, y \) are calculated from the parametric equations. When these points are plotted on an \( xy \) plane they trace out a curve. The Cartesian equation of this curve is obtained by eliminating the parameter \( t \) from the parametric equations. For example, consider the curve:

\[ \begin{align*}
  x &= 2 \cos t \\
  y &= 2 \sin t
\end{align*} \quad 0 \leq t \leq 2\pi. \]

We can eliminate the \( t \) variable in an obvious way - square each parametric equation and then add:

\[ x^2 + y^2 = 4 \cos^2 t + 4 \sin^2 t = 4 \quad \therefore \quad x^2 + y^2 = 4 \]

which we recognise as the standard equation of a \textit{circle} with centre at \((0, 0)\) with radius 2.

In a similar fashion the parametric equations

\[ \begin{align*}
  x &= 2t \\
  y &= 4t^2
\end{align*} \quad -\infty < t < \infty \]

describe a \textit{parabola}. This follows since, eliminating the parameter \( t \):

\[ t = \frac{x}{2} \quad \therefore \quad y = 4 \left( \frac{x^2}{4} \right) \quad \text{so} \quad y = x^2 \]

which we recognise as the standard equation of a parabola.

The question we wish to address in this Section is 'how do we obtain the derivative \( \frac{dy}{dx} \) if a curve is given in parametric form?' To answer this we note the key result in this area:

\[ \textbf{Key Point 12} \]

**Parametric Differentiation**

If \( x = h(t) \) and \( y = g(t) \) then

\[ \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \]

We note that this result allows the determination of \( \frac{dy}{dx} \) without the need to find \( y \) as an explicit function of \( x \).
Example 13

Determine the equation of the tangent line to the semicircle with parametric equations
\[ x = \cos t \quad y = \sin t \quad 0 \leq t \leq \pi \]
at \( t = \pi/4 \).

Solution

The semicircle is drawn in Figure 9. We have also drawn the tangent line at \( t = \pi/4 \) (or, equivalently, at \( x = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad y = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \)).

Now
\[ \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\cos t}{-\sin t} = -\cot t. \]

Thus at \( t = \pi/4 \) we have \( \frac{dy}{dx} = -\cot \left( \frac{\pi}{4} \right) = -1. \)

The equation of the tangent line is
\[ y = mx + c \]
where \( m \) is the gradient of the line and \( c \) is a constant.

Clearly \( m = -1 \) (since, at the point \( P \) the line and the circle have the same gradient).

To find \( c \) we note that the line passes through the point \( P \) with coordinates \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \). Hence
\[ \frac{1}{\sqrt{2}} = (-1) \frac{1}{\sqrt{2}} + c \quad \therefore \quad c = \frac{2}{\sqrt{2}} \]

Finally,
\[ y = -x + \frac{2}{\sqrt{2}} \]
is the equation of the tangent line at the point in question.
We should note, before proceeding, that a derivative with respect to the parameter $t$ is often denoted by a ‘dot’. Thus
\[
\frac{dx}{dt} = \dot{x}, \quad \frac{dy}{dt} = \dot{y}, \quad \frac{d^2x}{dt^2} = \ddot{x} \quad \text{etc.}
\]

**Task**

Find the value of $\frac{dy}{dx}$ if $x = 3t$, $y = t^2 - 4t + 1$.

Check your result by finding $\frac{dy}{dx}$ in the normal way.

First find $\frac{dx}{dt}$, $\frac{dy}{dt}$:

**Your solution**

\[
\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 2t - 4
\]

**Answer**

\[
\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 2t - 4
\]

Now obtain $\frac{dy}{dx}$:

**Your solution**

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 4}{3} = \frac{2}{3}t - \frac{4}{3}
\]

**Answer**

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 4}{3} = \frac{2}{3}t - \frac{4}{3}
\]

or, using the ‘dot’ notation
\[
\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2t - 4}{3} = \frac{2}{3}t - \frac{4}{3}
\]

Now find $y$ explicitly as a function of $x$ by eliminating $t$, and so find $\frac{dy}{dx}$ directly:

**Your solution**

\[
t = \frac{x}{3} \quad \therefore \quad y = \frac{x^2}{9} - \frac{4x}{3} + 1.
\]

Finally:
\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2x}{9} - \frac{4}{3}}{\frac{3}{3}} = \frac{2}{3}t - \frac{4}{3}.
\]
Find the value of \( \frac{dy}{dx} \) at \( t = 2 \) if \( x = 3t - 4 \sin \pi t, \quad y = t^2 + t \cos \pi t, \quad 0 \leq t \leq 4 \)

First find \( \frac{dx}{dt}, \frac{dy}{dt} \):

Your solution

\[
\frac{dx}{dt} = 3 - 4 \pi \cos \pi t \quad \frac{dy}{dt} = 2t + \cos \pi t - \pi t \sin \pi t
\]

Now obtain \( \frac{dy}{dx} \):

Your solution

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t + \cos \pi t - \pi t \sin \pi t}{3 - 4 \pi \cos \pi t}
\]

or, using the dot notation,

\[
\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2t + \cos \pi t - \pi t \sin \pi t}{3 - 4 \pi \cos \pi t}
\]

Finally, substitute \( t = 2 \) to find \( \frac{dy}{dx} \) at this value of \( t \).

Your solution

\[
\left| \frac{dy}{dx} \right|_{t=2} = \frac{4 + 1}{3 - 4 \pi} = \frac{5}{3 - 4 \pi} = -0.523
\]
2. Higher derivatives

Having found the first derivative \( \frac{dy}{dx} \) using parametric differentiation we now ask how we might determine the second derivative \( \frac{d^2y}{dx^2} \).

By definition:

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right)
\]

But

\[
\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \quad \text{and so} \quad \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{\dot{y}}{\dot{x}} \right)
\]

Now \( \frac{\dot{y}}{\dot{x}} \) is a function of \( t \) so we can change the derivative with respect to \( x \) into a derivative with respect to \( t \) since

\[
\frac{d}{dx} \left( \frac{\dot{y}}{\dot{x}} \right) = \frac{\frac{dt}{dx}}{\dot{x}^2}
\]

from the function of a function rule (Key Point 11 in Section 11.5).

But, differentiating the quotient \( \frac{\dot{y}}{\dot{x}} \), we have

\[
\frac{d}{dt} \left( \frac{\dot{y}}{\dot{x}} \right) = \frac{\dot{x} \ddot{y} - \ddot{x} \dot{y}}{\dot{x}^3} \quad \text{and} \quad \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{\dot{x}}
\]

so finally:

\[
\frac{d^2y}{dx^2} = \frac{\dot{x} \ddot{y} - \ddot{x} \dot{y}}{\dot{x}^3}
\]

Key Point 13

If \( x = h(t) \), \( y = g(t) \) then the first and second derivatives of \( y \) with respect to \( x \) are:

\[
\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{\dot{x} \ddot{y} - \ddot{x} \dot{y}}{\dot{x}^3}
\]
Example 14

If the equations of a curve are \( x = 2t, \ y = t^2 - 3 \), determine \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \).

Solution

Here \( \dot{x} = 2, \ \dot{y} = 2t \) \( \therefore \) \( \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2t}{2} = t \).

Also \( \ddot{x} = 0, \ \ddot{y} = 2 \) \( \therefore \) \( \frac{d^2y}{dx^2} = \frac{2(2) - 2t(0)}{(2)^3} = \frac{1}{2} \).

These results can easily be checked since \( t = \frac{x}{2} \) and \( y = t^2 - 3 \) which imply \( y = \frac{x^2}{4} - 3 \). Therefore the derivatives can be obtained directly: \( \frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2} \) and \( \frac{d^2y}{dx^2} = \frac{1}{2} \).

Exercises

1. For the following sets of parametric equations find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \).
   
   (a) \( x = 3t^2 \quad y = 4t^3 \) 
   
   (b) \( x = 4 - t^2 \quad y = t^2 + 4t \) 
   
   (c) \( x = t^2e^t \quad y = t \)

2. Find the equation of the tangent line to the curve

   \( x = 1 + 3\sin t \quad y = 2 - 5\cos t \) at \( t = \frac{\pi}{6} \)

Answers

1. (a) \( \frac{dy}{dx} = 2t, \ \frac{d^2y}{dx^2} = \frac{1}{3}t \) (b) \( \frac{dy}{dx} = -1 - \frac{2}{t}, \ \frac{d^2y}{dx^2} = -\frac{1}{t^3} \)
   
   (c) \( \frac{dy}{dx} = \frac{e^{-t}}{2t + t^2}, \ \frac{d^2y}{dx^2} = -\frac{e^{-2t}(t^2 + 4t + 2)}{(t + 2)^3t^3} \)

2. \( \dot{x} = 3\cos t \quad \dot{y} = +5\sin t \)

\( \therefore \) \( \frac{dy}{dx} = \frac{5}{3}\tan t \) \( \therefore \) \( \left. \frac{dy}{dx} \right|_{t=\pi/6} = \frac{5}{3}\tan \frac{\pi}{6} = \frac{5}{3}\times\frac{1}{\sqrt{3}} = \frac{5\sqrt{3}}{9} \)

The equation of the tangent line is \( y = mx + c \) where \( m = \frac{5\sqrt{3}}{9} \).

The line passes through the point \( x = 1 + 3\sin \frac{\pi}{6} = 1 + \frac{3}{2}, \ y = 2 - 5\frac{\sqrt{3}}{2} \) and so

\( 2 - 5\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{9}(1 + \frac{3}{2}) + c \) \( \therefore \) \( c = 2 - \frac{35\sqrt{3}}{9} \)